Closing Tue: 14.1,14.3(1),14.3(2)
Closing Thu: 14.4, 14.7(1)
Office Hours: 1:30-3:00pm, Smith 309

14.3/14.4 Partial Der. & Tangent Planes

Note: A variable can be treated as

1. A constant
2. An independent variable (input)
3. A dependent variable (output)

Entry Task: Find the derivatives

a) $y = f(x) = x^2 e^x$, $\frac{dy}{dx} = ??$

b) An object's motion (x,y) = (x(t),y(t)) satisfies y = x^2 for all times. $\frac{dy}{dt} = ??$

c) $x^2 + y^3 = 1$, $\frac{dy}{dx} = ??$

d) $z = x^2 + y^3 e^{6y} - 5xy^4$

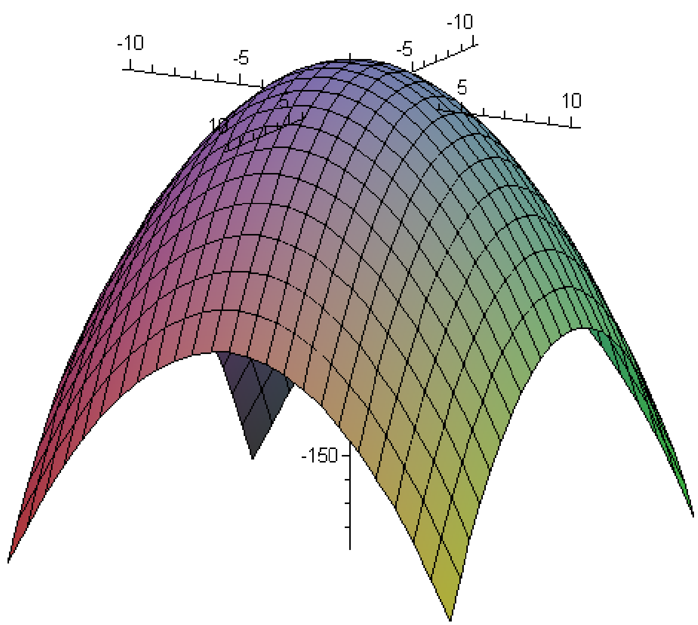
$\frac{\partial z}{\partial x} =$

$\frac{\partial z}{\partial y} =$

e) $x^2 + y^2 - z^2 = 1$, $\frac{\partial z}{\partial x} = ??$

Graphical Interpretation of Partial Der:

Pretend you are skiing on the surface $z = f(x, y) = 15 - x^2 - y^2$.

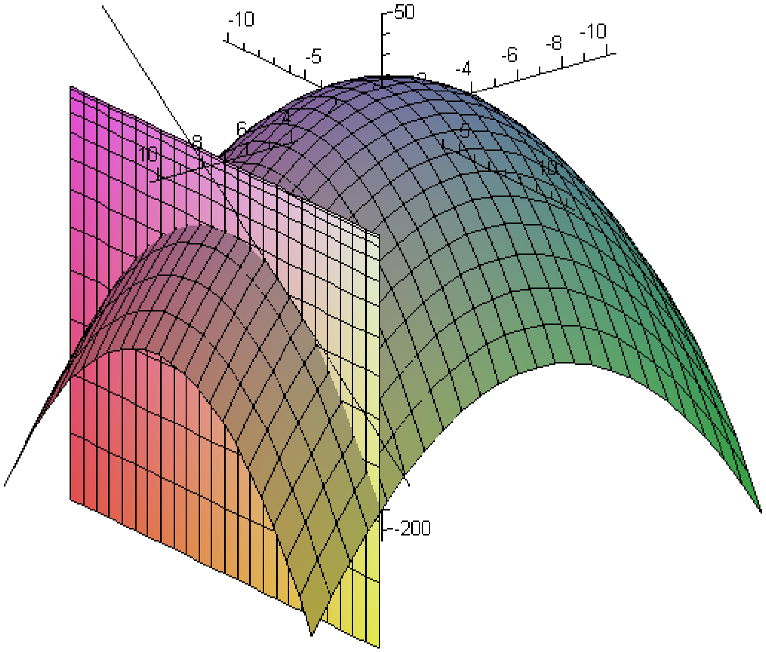


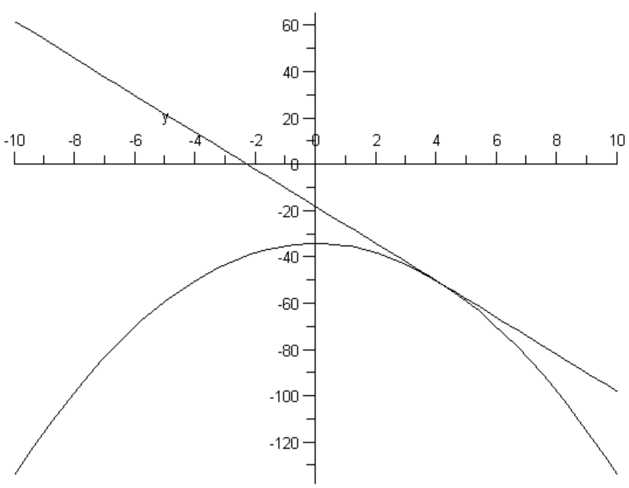
Exercise:

1. Find $f_x(x, y)$ and $f_y(x, y)$
2. Assume you are standing on the point on the surface corresponding to (x,y) = (4,7). Compute:
   i) $f(4,7) =$
   ii) $f_x(4,7) =$
   iii) $f_y(4,7) =$

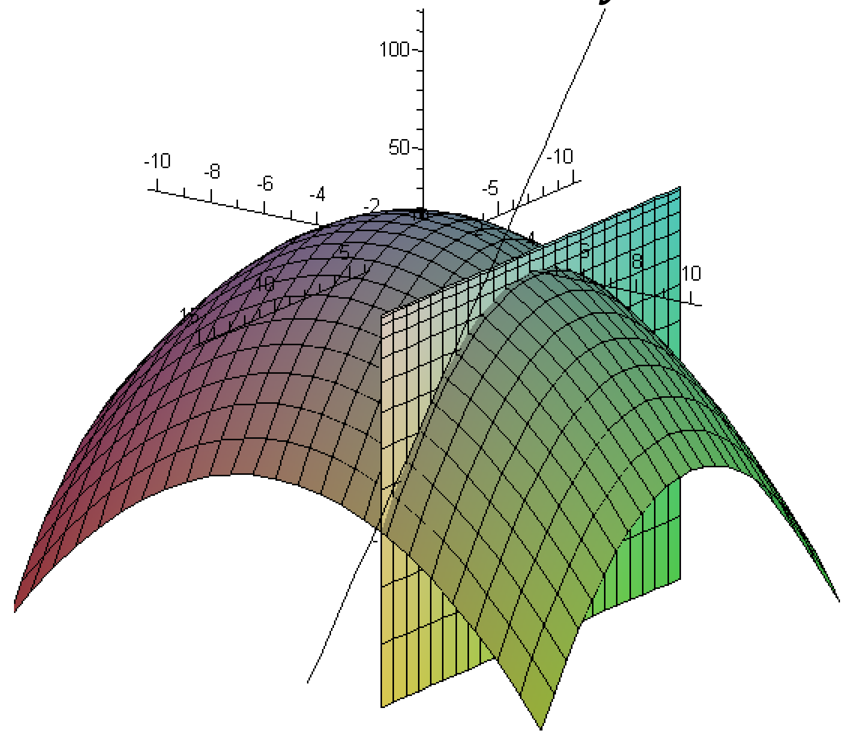
   What do these three numbers represent?

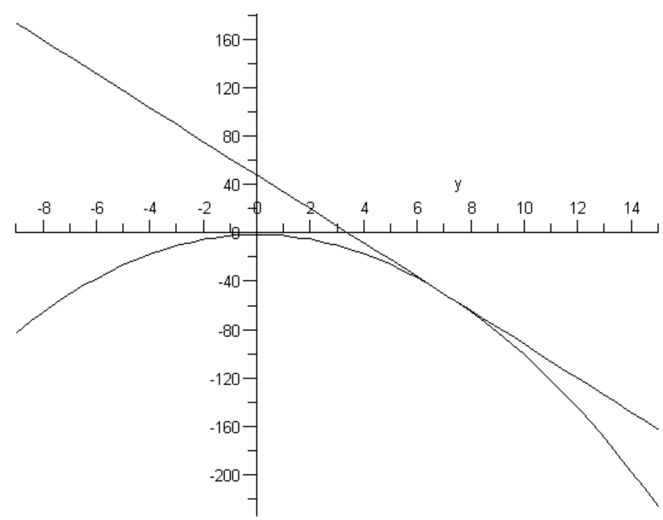
The plane x = 4 intersecting the surface $z = 15 - x^2 - y^2$.





The plane y = 7 intersecting the surface $z = 15 - x^2 - y^2$.





Second Partial Derivatives

Concavity in *x*-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}(x, y)$$

Concavity in *y*-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}(x, y)$$

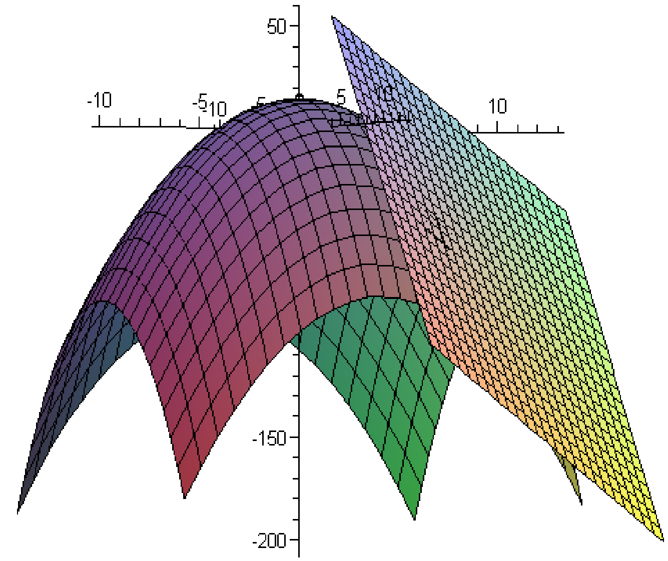
Mixed Partials:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}(x, y)$$

Example: Find all second partials for
$z = f(x, y) = x^4 + 3x^2y^3 + y^5$

14.4 Tangent Planes (linear approx.)



The <u>tangent plane</u> to a surface at a point is the plane that contains all tangent lines at that point. It is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example: Find the tangent plane to

$z = f(x,y) = 15 - x^2 - y^2$ at (7,4)

Recall:

$f(4,7) =$

$f_x(4,7) =$

$f_y(4,7) =$

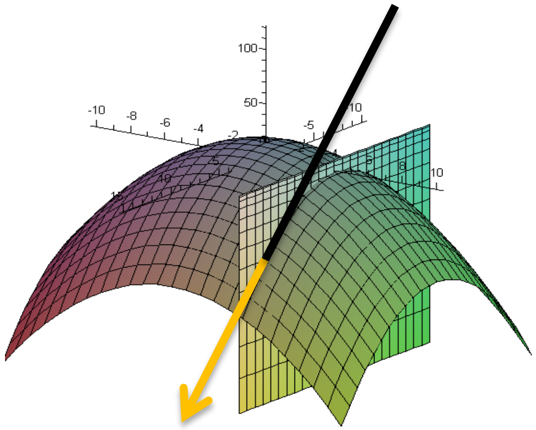
Derivation of Tangent Plane

We are trying to find the equation for a plane. The plane goes thru x = 7, y = 4, z = -50. Now we need a normal vector.

Note:

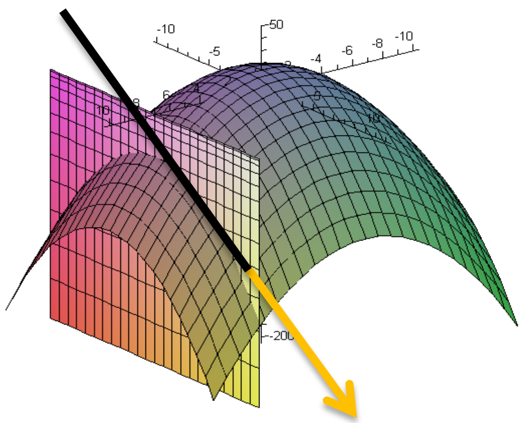
$f_x(x,y) = -2x$

$f_x(7,4) = -14$



$f_y(x,y) = -2y$

$f_y(7,4) = -8$



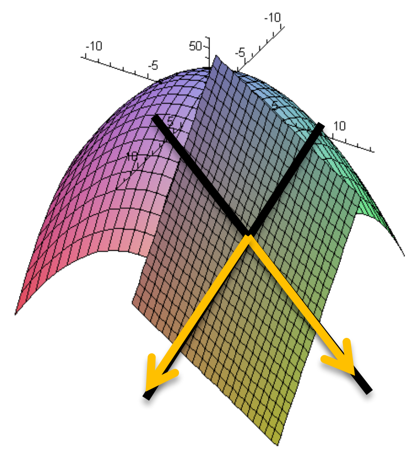
Thus, we can get two vectors that are parallel to the plane:

$$<1,0,f_x(x_0,y_0)> = <1,0,-14>$$

$$<0,1,f_y(x_0,y_0)> = <0,1,-8>$$

So a normal vector is given by

$<1,0,-14> \times <0,1,-8> = <14, 8, 1>$



Tangent Plane:

$$14(x-7) + 8(y-4) + (z+50) = 0$$

Which we rewrite as:

$$z + 50 = -14(x-7) - 8(y-4)$$

In general, for $z = f(x,y)$ at (x_0, y_0) by:

1. $z_0 = f(x_0, y_0)$ = height.
2. $\langle 1,0, f_x(x_0, y_0)\rangle$ = 'a tangent in *x*-dir.'
   $\langle 0,1, f_y(x_0, y_0)\rangle$ = 'a tangent in *y*-dir.'
3. Normal to surface:
   $\langle 1,0, f_x(x_0, y_0)\rangle \times \langle 0,1, f_y(x_0, y_0)\rangle$
   $= \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$

Tangent Plane:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which we typically write as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example: Find the tangent plane for

$$f(x,y) = x^2 + 3y^2x - y^3$$

at $(x,y) = (2,1)$.

Applications of the Tangent Planes

1. *Linear Approximation*:
   "Near" the points (x_0,y_0) the tangent plane and surface z-values are close.
2. *Differentials*: Same idea in terms of differences.
   Actual changes on surface are:
   $\Delta x=x-x_0$, $\Delta y=y-y_0$, $\Delta z=f(x,y) - f(x_0,y_0)$
   Approx changes on tangent plane:
   $dx= x - x_0$, $dy = y - y_0$, $dz = z - z_0$

Example: Find the linear approximation and total differential for

$$f(x,y) = x^2 + 3y^2x - y^3$$

at $(x,y) = (2,1)$.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x, y) = z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

Concepts:
$f(x, y) \approx L(x, y)$ for $(x, y) \approx (x_0, y_0)$, and
$\Delta z \approx dz$ for $(x, y) \approx (x_0, y_0)$

HW: Use differentials to estimate the amount of metal in a closed cylindrical can that is 18 cm high and 8 cm in diameter if the metal in the top and the bottom is 0.2 cm thick and the metal in the sides is 0.05 cm thick. (Round your answer to two decimal places.)

Hint: *Estimate change in volume for a cylinder,* dV = ???

Let r = radius and h = height, then

$$V = \pi r^2 h$$

Total differential:

$$dV = 2\pi r h \, dr + \pi r^2 \, dh$$

now plug in numbers:

$$h = 18, \; r = 4, \; dh = 0.4, \; dr = 0.05$$